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# Strange quark mass from pseudoscalar sum rule with $O(\alpha_s^4)$ accuracy

K.G. Chetyrkin[1,a], A. Khodjamirian[2,b]

[1] Institut für Theoretische Teilchenphysik, Universität Karlsruhe, 76128 Karlsruhe, Germany
[2] Theoretische Physik 1, Fachbereich Physik, Universität Siegen, 57068 Siegen, Germany

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**Abstract.** We include the new, five-loop, $O(\alpha_s^4)$ correction into the QCD sum rule used for the $s$-quark mass determination. The pseudoscalar Borel sum rule is taken as a study case. The OPE for the correlation function with $N^4LO$, $O(\alpha_s^4)$ accuracy in the perturbative part, and with dimension $d \leq 6$ operators reveals a good convergence. We observe a significant improvement of stability of the sum rule with respect to the variation of the renormalization scale after including the $O(\alpha_s^4)$ correction. We obtain the interval $\overline{m}_s(2\,\mathrm{GeV}) = 105 \pm 6 \pm 7\,\mathrm{MeV}$, which exhibits about 2 MeV increase of the central value, if the $O(\alpha_s^4)$ terms are removed.

## 1 Introduction

A precise determination of the strange quark mass $m_s$ is extremely important for various tests of Standard Model. A reach variety of approaches is used to evaluate this fundamental parameter in QCD. Recently, the first unquenched lattice QCD determinations became available [1–6]. In addition, ChPT provides rather accurate ratios of strange and nonstrange quark masses [7, 8]. Furthermore, one evaluates $m_s$, combining the operator product expansion (OPE) of various correlation functions for strangeness-changing quark currents with dispersion relations. These methods include model-independent bounds [9], QCD analyses of hadronic $\tau$ decays (see [10–14] for the latest results), as well as different versions of QCD sum rules [15] and related finite-energy sum rules (FESR) [16]. The most recent sum rule determinations of $m_s$ in the channels of scalar, pseudoscalar and vector currents are presented in [17–19], respectively, references to earlier analyses can be found in reviews [20, 21] (see also [22]). The estimated accuracy of these results is 15%–30%. To achieve a better precision, one has to calculate higher orders in OPE of the correlation functions and gain a better control over potentially important nonperturbative corrections beyond OPE (the so called "direct instantons"). Furthermore, more accurate data for the inputs in hadronic spectral functions and a better assessment of the quark-hadron duality are needed.

In this paper we concentrate on the QCD sum rules used to evaluate the strange quark mass and make one further step to improve the accuracy of this determination by including the $N^4LO$ perturbative QCD corrections of $O(\alpha_s^4)$ into the sum rule. The $O(\alpha_s^4)$, five-loop contribution has recently been calculated for the correlator of the scalar quark currents in [23] and can be equally well used for both scalar and pseudoscalar sum rules. As a study case, we choose the pseudoscalar version of the standard Borel sum rule. For the hadronic spectral function we employ the three-resonance ansatz worked out in [18].

We find that both OPE for the correlation function and the resulting sum rule reveal a good numerical convergence in powers of $\alpha_s$. The new $O(\alpha_s^4)$ correction to the sum rule has a naturally small influence, resulting in about 2 MeV decrease of the s-quark mass $\overline{m}_s$ (in $\overline{MS}$ scheme) determined with $O(\alpha_s^3)$ accuracy. Importantly, after including the $O(\alpha_s^4)$ correction, we observe a significant improvement in the stability of the extracted value of $\overline{m}_s$ with respect to the renormalization scale variation in the sum rule. With $O(\alpha_s^4)$ accuracy we obtain the interval

$$\overline{m}_s(2\,\mathrm{GeV}) = \left(105 \pm 6\Big|_{\mathrm{param}} \pm 7\Big|_{\mathrm{hadr}}\right)\mathrm{MeV}\,, \qquad (1)$$

where the estimated uncertainties from the sum rule parameters and hadronic inputs are shown separately and will be explained below.

In what follows, after a brief recapitulation of the pseudoscalar Borel sum rule in Sect. 2, we present in Sect. 3 the QCD OPE expressions for the underlying correlation function, including the new $O(\alpha_s^4)$ terms in the perturbative part. The Borel transform and the imaginary part of the correlation function are also given. In Sect. 4 we turn to the numerical analysis of the sum rule and obtain the interval (1). Section 5 contains the concluding discussion.

[a] *Permanent address*: Institute for Nuclear Research, Russian Academy of Sciences, Moscow 117312, Russia.
[b] e-mail: khodjam@hep.physik.uni-siegen.de

## 2 Pseudoscalar sum rule

We consider the correlation function:

$$\Pi^{(5)}(q^2) = \mathrm{i}\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x}\langle 0|T\left\{j_5^{(s)}(x)j_5^{(s)\dagger}(0)\right\}|0\rangle \quad (2)$$

of the two pseudoscalar strangeness-changing quark currents, defined as the divergences of the corresponding axial-vector currents:

$$j_5^{(s)} = \partial^\mu(\bar{s}\gamma_\mu\gamma_5 q) = (m_s + m_q)\bar{s}\mathrm{i}\gamma_5 q. \quad (3)$$

For definiteness, the light quark $q = u$ is taken.

The Borel sum rule is obtained following the standard SVZ method [15] and is based on the (double-subtracted) dispersion relation for $\Pi^{(5)}(q^2)$. This relation is more conveniently written in a form of the second derivative:

$$\Pi^{(5)''}(q^2) \equiv \frac{\mathrm{d}^2}{\mathrm{d}(q^2)^2}\Pi^{(5)}(q^2) = \frac{2}{\pi}\int\limits_0^\infty \mathrm{d}s \frac{\mathrm{Im}\,\Pi^{(5)}(s)}{(s-q^2)^3}\,. \quad (4)$$

After Borel transformation one obtains[1]:

$$\Pi^{(5)''}(M^2) \equiv \mathcal{B}_{M^2}\left[\Pi^{(5)''}(q^2)\right] = \frac{1}{\pi M^4}\int\limits_0^\infty \mathrm{d}s\, e^{-s/M^2}\mathrm{Im}\,\Pi^{(5)}(s)\,. \quad (5)$$

The l.h.s. of the above relation is calculated in QCD at large $M^2 \gg \Lambda_{QCD}^2$ in a form of OPE (perturbative and condensate expansion), in powers of $\alpha_s$ and $m_s/M$, and up to a certain dimension of vacuum condensates. In r.h.s. the hadronic spectral density $\rho^{(5)}_{\mathrm{hadr}}(s) = (1/\pi)\mathrm{Im}\,\Pi^{(5)}(s)$ is substituted. At $s < s_0$ , where $s_0$ is some effective threshold, the function $\rho^{(5)}_{\mathrm{hadr}}(s)$ includes kaon and its excitations. The rest of the hadronic dispersion integral at $s > s_0$ is approximated using quark-hadron duality, $\rho^{(5)}_{\mathrm{hadr}}(s) \simeq \rho^{(5)}_{OPE}(s)$, with the spectral function calculated from OPE: $\rho^{(5)}_{\mathrm{OPE}}(s) = (1/\pi)\mathrm{Im}\,[\Pi^{(5)}(s)]_{\mathrm{OPE}}$. The final form of the sum rule is:

$$M^4[\Pi^{(5)''}(M^2)]_{\mathrm{OPE}} = \int\limits_0^{s_0} \mathrm{d}s\, e^{-s/M^2}\rho^{(5)}_{\mathrm{hadr}}(s) + \int\limits_{s_0}^\infty \mathrm{d}s\, e^{-s/M^2}\rho^{(5)}_{\mathrm{OPE}}(s)\,. \quad (6)$$

In the following we discuss both parts of this equation in detail.

## 3 OPE results to $O(\alpha_s^4)$

In this section we present the expressions for $[\Pi^{(5)''}(q^2)]_{\mathrm{OPE}}$ and, correspondingly, for $[\Pi^{(5)''}(M^2)]_{\mathrm{OPE}}$ and $\rho^{(5)}_{\mathrm{OPE}}(s)$ determining the QCD input in the sum rule (6). The OPE for $[\Pi^{(5)''}(q^2)]_{\mathrm{OPE}}$ goes over powers of $(1/|q|)^{d+2}$ ordered by the dimension $d = 0, 2, 4, 6$. The OPE terms with $d > 6$ are neglected, while already the $d = 6$ contribution is very small in the working region of the variables $Q^2$ and $M^2$.

The $d = 0, 2$ terms of OPE originate from the perturbative part of the correlation function. The expansion in quark-gluon coupling up to four loops, that is, up to $O(\alpha_s^3)$, can be taken from [24–26]. The new $O(\alpha_s^4)$ terms are obtained in [23]. Putting them together, we obtain:

$$[\Pi^{(5)''}(Q^2)]^{(d=0,2)}_{\mathrm{OPE}} = \frac{3(m_s+m_u)^2}{8\pi^2Q^2}\left\{1+\sum_i \bar{d}_{0,i}\,a_s^i - 2\frac{m_s^2}{Q^2}\left(1+\sum_i \bar{d}_{2,i}\,a_s^i\right)\right\}, \quad (7)$$

where $Q^2 = -q^2$, and the coefficients multiplying the powers of the quark-gluon coupling $a_s = \alpha_s(\mu)/\pi$ are

$$\bar{d}_{0,1} = \frac{11}{3} - 2\,l_Q,\ \ \bar{d}_{0,2} = \frac{5071}{144} - \frac{35}{2}\zeta_3 - \frac{139}{6}\,l_Q + \frac{17}{4}\,l_Q^2, \quad (8)$$

$$\bar{d}_{0,3} = \frac{1995097}{5184} - \frac{1}{36}\pi^4 - \frac{65869}{216}\zeta_3 + \frac{715}{12}\zeta_5 - \frac{2720}{9}\,l_Q + \frac{475}{4}\zeta_3\,l_Q + \frac{695}{8}\,l_Q^2 - \frac{221}{24}\,l_Q^3, \quad (9)$$

$$\begin{aligned}\bar{d}_{0,4} = {}& \frac{2361295759}{497664} - \frac{2915}{10368}\pi^4 - \frac{25214831}{5184}\zeta_3 + \frac{192155}{216}\zeta_3^2 + \frac{59875}{108}\zeta_5 - \frac{625}{48}\zeta_6 - \frac{52255}{256}\zeta_7 \\ &+ l_Q\left[-\frac{43647875}{10368} + \frac{1}{18}\pi^4 + \frac{864685}{288}\zeta_3 - \frac{24025}{48}\zeta_5\right] \\ &+ l_Q^2\left[\frac{1778273}{1152} - \frac{16785}{32}\zeta_3\right] + l_Q^3\left[-\frac{79333}{288}\right] + l_Q^4\left[\frac{7735}{384}\right],\end{aligned} \quad (10)$$

$$\bar{d}_{2,1} = \frac{28}{3} - 4\,l_Q,\ \ \bar{d}_{2,2} = \frac{8557}{72} - \frac{77}{3}\zeta_3 - \frac{147}{2}\,l_Q + \frac{25}{2}\,l_Q^2\,, \quad (11)$$

including the new result for $\bar{d}_{0,4}$. Here $l_Q = \log\frac{Q^2}{\mu^2}$, and $\zeta_n \equiv \zeta(n)$ is the Riemann's Zeta-function. The coupling $a_s$ and the quark masses $m_s$ and $m_u$ are all taken in $\overline{MS}$ scheme at the renormalization scale $\mu$. We have neglected the light-quark mass $m_u$, except in the overall factors. Note also that in the subleading $d = 2$, $O(m_s^4)$ terms of the above expansion, the currently achieved $O(\alpha_s^2)$ accuracy is quite sufficient.

The contributions with $d = 4, 6$ in the correlation function originate both from nonperturbative (condensate) terms and from $O(m_s^6)$ corrections, and we use the known

[1] Here we use the following normalization convention: $\mathcal{B}_{M^2}[1/(a-q^2)] = e^{-a/M^2}$.

expressions [26, 27]

$$[\Pi^{(5)''}(q^2)]^{(d=4,6)}_{\rm OPE} = \frac{(m_s+m_u)^2}{Q^6}\left\{-2m_s\langle\bar{u}u\rangle\left(1+a_s\left(\frac{23}{3}-2l_Q\right)\right) -\frac{1}{9}I_G\left(1+a_s\left(\frac{121}{18}-2l_Q\right)\right)+I_s\left(1+a_s\left(\frac{64}{9}-2l_Q\right)\right) -\frac{3}{7\pi^2}m_s^4\left(\frac{1}{a_s}+\frac{155}{24}-\frac{15}{4}l_Q\right)+\frac{I_6}{Q^2}\right\}, \quad (12)$$

where

$$I_s = m_s\langle\bar{s}s\rangle+\frac{3}{7\pi^2}m_s^4\left(\frac{1}{a_s}-\frac{53}{24}\right) \quad (13)$$

and

$$I_G = -\frac{9}{4}\langle\frac{\alpha_s}{\pi}G^2\rangle\left(1+\frac{16}{9}a_s\right)+4a_s\left(1+\frac{91}{24}a_s\right)m_s\langle\bar{s}s\rangle +\frac{3}{4\pi^2}\left(1+\frac{4}{3}a_s\right)m_s^4 \quad (14)$$

are the vacuum expectation values of two RG-invariant combinations of dimension 4 containing quark and gluon condensate densities (for details and explanation see [26–29]). Finally,

$$I_6 = -3m_s\langle\bar{u}uG\rangle-\frac{32}{9}\pi^2a_s\Big(\langle\bar{u}u\rangle^2+\langle\bar{s}s\rangle^2-9\langle\bar{u}u\rangle\langle\bar{s}s\rangle\Big) \quad (15)$$

is the combination of dimension-6 contributions of the quark-gluon and 4-quark condensates (the vacuum saturation is assumed for the latter).

The Borel transform of (7) and (12) is given by

$$[\Pi^{(5)''}(M^2)]^{(d=0,2)}_{\rm OPE} = \frac{3(m_s+m_u)^2}{8\pi^2}\left\{1+\sum_i\bar{b}_{0,i}\,a_s^i-2\frac{m_s^2}{M^2}\left(1+\sum_i\bar{b}_{2,i}\,a_s^i\right)\right\}, \quad (16)$$

where $l_M=\log\frac{M^2}{\mu^2}$ and the coefficients are

$$\bar{b}_{0,1}=\frac{11}{3}+2\,\gamma_E-2\,l_M, \quad (17)$$

$$\bar{b}_{0,2}=\frac{5071}{144}+\frac{139}{6}\gamma_E+\frac{17}{4}\gamma_E^2-\frac{17}{24}\pi^2-\frac{35}{2}\zeta_3-\frac{139}{6}l_M -\frac{17}{2}\gamma_E\,l_M+\frac{17}{4}l_M^2, \quad (18)$$

$$\bar{b}_{0,3}=\frac{1995097}{5184}+\frac{2720}{9}\gamma_E+\frac{695}{8}\gamma_E^2+\frac{221}{24}\gamma_E^3-\frac{695}{48}\pi^2 -\frac{221}{48}\gamma_E\pi^2-\frac{1}{36}\pi^4-\frac{61891}{216}\zeta_3-\frac{475}{4}\gamma_E\,\zeta_3+\frac{715}{12}\zeta_5 +l_M\left[-\frac{2720}{9}-\frac{695}{4}\gamma_E-\frac{221}{8}\gamma_E^2+\frac{221}{48}\pi^2+\frac{475}{4}\zeta_3\right] +l_M^2\left[\frac{695}{8}+\frac{221}{8}\gamma_E\right]-\frac{221}{24}l_M^3, \quad (19)$$

$$b_{0,4}=\frac{2361295759}{497664}+\frac{43647875}{10368}\gamma_E+\frac{1778273}{1152}\gamma_E^2 +\frac{79333}{288}\gamma_E^3+\frac{7735}{384}\gamma_E^4-\frac{1778273}{6912}\pi^2-\frac{79333}{576}\gamma_E\pi^2 -\frac{7735}{384}\gamma_E^2\pi^2+\frac{2263}{41472}\pi^4-\frac{1}{18}\gamma_E\pi^4-\frac{22358843}{5184}\zeta_3 -\frac{818275}{288}\gamma_E\,\zeta_3-\frac{16785}{32}\gamma_E^2\,\zeta_3+\frac{5595}{64}\pi^2\,\zeta_3 +\frac{192155}{216}\zeta_3^2+\frac{59875}{108}\zeta_5+\frac{24025}{48}\gamma_E\,\zeta_5-\frac{625}{48}\zeta_6 -\frac{52255}{256}\zeta_7 +l_M\Big[-\frac{43647875}{10368}-\frac{1778273}{576}\gamma_E-\frac{79333}{96}\gamma_E^2 -\frac{7735}{96}\gamma_E^3+\frac{79333}{576}\pi^2+\frac{7735}{192}\gamma_E\pi^2+\frac{1}{18}\pi^4 +\frac{818275}{288}\zeta_3+\frac{16785}{16}\gamma_E\,\zeta_3-\frac{24025}{48}\zeta_5\Big] +l_M^2\Big[\frac{1778273}{1152}+\frac{79333}{96}\gamma_E+\frac{7735}{64}\gamma_E^2-\frac{7735}{384}\pi^2 -\frac{16785}{32}\zeta_3\Big] +l_M^3\left[-\frac{79333}{288}-\frac{7735}{96}\gamma_E\right]+l_M^4\left[\frac{7735}{384}\right], \quad (20)$$

$$\bar{b}_{2,1}=\frac{16}{3}+4\,\gamma_E-4\,l_M, \quad (21)$$

$$\bar{b}_{2,2}=\frac{5065}{72}+\frac{97}{2}\gamma_E+\frac{25}{2}\gamma_E^2-\frac{25}{12}\pi^2-\frac{77}{3}\zeta_3-\frac{97}{2}l_M -25\,\gamma_E\,l_M+\frac{25}{2}l_M^2, \quad (22)$$

and, respectively,

$$[\Pi^{(5)''}(M^2)]^{(d=4,6)}_{\rm OPE} = \frac{(m_s+m_u)^2}{2M^4}\left\{-2m_s\langle\bar{u}u\rangle\left(1+a_s\left(\frac{14}{3}+2\gamma_E-2l_M\right)\right) -\frac{1}{9}I_G\left(1+a_s\left(\frac{67}{18}+2\gamma_E-2l_M\right)\right) +I_s\left(1+a_s\left(\frac{37}{9}+2\gamma_E-2l_M)\right)\right) -\frac{3}{7\pi^2}m_s^4\left(\frac{1}{a_s}+\frac{5}{6}+\frac{15}{4}\gamma_E-\frac{15}{4}l_M\right)+\frac{I_6}{3M^2}\right\}. \quad (23)$$

In addition, we need the imaginary part of the correlation function calculated with the same $\alpha_s^4$ accuracy as (16) and

(23):

$$\rho^{(5)}_{\rm OPE}(s) = \frac{1}{\pi}{\rm Im}\Pi^{(5)}(s)$$
$$= \frac{3(m_s+m_u)^2}{8\pi^2}\,s \times \left\{1+\sum_i \tilde{r}_{0,i}\,a_s^i - 2\frac{m_s^2}{s}\left(1+\sum_i \tilde{r}_{2,i}\,a_s^i\right)\right\}$$
$$+\frac{m_s^2(s)}{s}\left\{\frac{45}{56\pi^2}m_s^4(s)+2a_s(s)m_s\langle\bar{u}u\rangle + \frac{a_s(s)}{9}I_G - a_s(s)I_s\right\}, \quad (24)$$

where $l_s = \log\frac{s}{\mu^2}$ and

$$\tilde{r}_{0,1} = \frac{17}{3} - 2\,l_s,$$
$$\tilde{r}_{0,2} = \frac{9631}{144} - \frac{17}{12}\pi^2 - \frac{35}{2}\zeta_3 - \frac{95}{3}\,l_s + \frac{17}{4}\,l_s^2, \quad (25)$$

$$\tilde{r}_{0,3} = \frac{4748953}{5184} - \frac{229}{6}\pi^2 - \frac{1}{36}\pi^4 - \frac{91519}{216}\zeta_3 + \frac{715}{12}\zeta_5 - \frac{4781}{9}\,l_s + \frac{221}{24}\pi^2\,l_s + \frac{475}{4}\zeta_3\,l_s + \frac{229}{2}\,l_s^2 - \frac{221}{24}\,l_s^3, \quad (26)$$

$$\tilde{r}_{0,4} = \frac{7055935615}{497664} - \frac{3008729}{3456}\pi^2 + \frac{19139}{5184}\pi^4 - \frac{46217501}{5184}\zeta_3 + \frac{5595}{32}\pi^2\zeta_3 + \frac{192155}{216}\zeta_3^2 + \frac{455725}{432}\zeta_5 - \frac{625}{48}\zeta_6 - \frac{52255}{256}\zeta_7 + l_s\left[-\frac{97804997}{10368} + \frac{51269}{144}\pi^2 + \frac{1}{18}\pi^4 + \frac{1166815}{288}\zeta_3 - \frac{24025}{48}\zeta_5\right] + l_s^2\left[\frac{3008729}{1152} - \frac{7735}{192}\pi^2 - \frac{16785}{32}\zeta_3\right] - \frac{51269}{144}\,l_s^3 + \frac{7735}{384}\,l_s^4, \quad (27)$$

$$\tilde{r}_{2,1} = \frac{16}{3} - 4\,l_s,$$
$$\tilde{r}_{2,2} = \frac{5065}{72} - \frac{25}{6}\pi^2 - \frac{77}{3}\zeta_3 - \frac{97}{2}\,l_s + \frac{25}{2}\,l_s^2. \quad (28)$$

The OPE expressions are valid at sufficiently large $Q^2 \gg \Lambda^2_{QCD}$ or, correspondingly, at large $M^2$. It is well known that in the spin zero (scalar and pseudoscalar) channels the breakdown of OPE is expected to occur at relatively large $Q^2 \simeq 1\,\mathrm{GeV}^2$, due to the presence of nonperturbative vacuum effects which are beyond the local condensate expansion [30, 31]. Models of the correlation function based on instanton ensembles, such as the instanton liquid model (ILM) [32, 33] allow to penetrate to smaller $Q^2$. A remedy used in previous analyses of pseudoscalar sum rules is to add to the OPE series an instanton correction calculated in ILM. As realized, e.g., in [18], at sufficiently large $M^2$, practically already at $M^2 > 2\,\mathrm{GeV}^2$ the ILM correction is small, hence we will avoid it by choosing $2\,\mathrm{GeV}^2$ as a lower limit of the Borel mass.

For the reader's convenience, the lengthy coefficients appearing in (7), (16) and (24) are made available (in computer-readable form) in [34].

## 4 Hadronic spectral density and the sum rule

The spectral function $\rho^{(5)}_{\rm hadr}(s)$ in (6) is a positive definite sum of all hadronic states with strangeness and $J^P = 0^-$, located below the threshold $\sqrt{s_0}$, above which $\rho^{(5)}_{\rm hadr}(s)$ is approximated by the OPE spectral density. Clearly, the larger is $s_0$, the smaller is the sensitivity of the sum rule to this quark-hadron duality ansatz.

The lowest hadronic state is the kaon. Using the standard definition of the kaon decay constant

$$\langle 0|\bar{s}\gamma_\mu\gamma_5 u|K^+(q)\rangle = iq_\mu f_K\,, \quad (29)$$

one obtains the relevant hadronic matrix element of the pseudoscalar current:

$$\langle 0|j_5^{(s)}|K^+(q)\rangle = f_K m_K^2\,, \quad (30)$$

so that the kaon contribution to the hadronic spectral density reads:

$$\rho^{(5)}_K(s) = f_K^2 m_K^4\,\delta(m_K^2 - s)\,. \quad (31)$$

The two heavier pseudoscalar resonances [35] are $K_1 = K(1460)$ and $K_2 = K(1830)$ with the masses $m_{K_1} = 1460\,\mathrm{MeV}$ and $m_{K_2} = 1830\,\mathrm{MeV}$ and total widths $\Gamma_{K_1} = 260\,\mathrm{MeV}$ and $\Gamma_{K_2} = 250\,\mathrm{MeV}$, respectively. These resonances are not yet well established, in particular, no experimental errors are attributed to their masses and widths. In any case, it seems plausible that the hadronic spectral density in the pseudoscalar channel with strangeness is dominated by the kaon and $K_{1,2}$ resonances, making this channel less complicated than the scalar channel where the strong $K\pi$ scattering in S-wave ($J^P = 0^+$) demands a dedicated analysis (see e.g., [17, 36]).

A detailed analysis of the hadronic part in the pseudoscalar sum rules (in both FESR and Borel versions) is presented in [18], employing the hadronic spectral density where the contributions of two resonances $K_{1,2}$ with finite widths are simply added to the ground-state term of the kaon. Here we adopt the same ansatz for the hadronic spectral density [2] in the sum rule (6):

$$\rho^{(5)}_{\rm hadr}(s) = f_K^2 m_K^4\,\delta(m_K^2 - s) + \sum_{i=1,2} f_{K_i}^2 m_{K_i}^4 B_{K_i}(s)\,, \quad (32)$$

[2] For a different hadronic ansatz including $K^*\pi$ state explicitly, see [37].

where $B_{K_i}(s)$ are the finite-width (Breit–Wigner type) replacements of the $\delta$-function in the spectral density for $K_{1,2}$:

$$\delta(m_{K_i}^2 - s) \to B_{K_i}(s) = \frac{1}{\pi}\left(\frac{\Gamma_{K_i} m_{K_i}}{(s-m_K^2)^2 + (\Gamma_{K_i} m_{K_i})^2}\right). \tag{33}$$

In [18] using FESR, the decay constants $f_{K_1}$ and $f_{K_2}$ of $K_1$ and $K_2$ resonances (defined similarly to $f_K$) were fitted. As anticipated from ChPT, small values, in the ballpark of 20–30 MeV for both $f_{K_1}$ and $f_{K_2}$ were obtained. On the other hand, due to the large mass multiplying these constants, the effects of $K_1$ and $K_2$ are quite noticeable in the hadronic part of the sum rule, hence, one has to avoid too large values of $M^2$. We will use the estimates of $f_{K_1,K_2}$ from [18] as hadronic inputs in our numerical analysis of (6).

Further improvements of the hadronic ansatz are possible, but they our beyond our scope here. In particular, it seems important to investigate the role of multiparticle states in the hadronic spectral function, starting from the two-particle states $K^*\pi$, $K\rho$. In [18] it is assumed that multiparticle effects are at least partially taken into account in the finite widths of $K_{1,2}$. One usually neglects the possible contributions of the nondiagonal transitions to the hadronic spectral function, e.g., intermediate states of the type $\langle 0|j_5^{(s)}|K\rangle\langle K|K^*\pi\rangle\langle K^*\pi|K_1\rangle\langle K_1|j_5^{(s)}|0\rangle$. The analysis of the light-quark vector channel ($J^P = 1^-$) without and with strangeness (see e.g., [38]) indicates that the effects of mixing between separate resonances via intermediate multiparticle states could be noticeable. Here, adopting the ansatz (32) we tacitly assume that the total widths of $K_{1,2}$ account for the dominant contributions of multiparticle states. In order to estimate the influence of this effect, we will also consider a version of the hadronic spectral density (32) with the total widths of $K_{1,2}$ set to zero, interpreting the difference of the result with and without the widths as a rough estimate of the uncertainty due to multiparticle hadronic states.

## 5 Inputs and numerical results

For the running of the strong coupling $a_s$ and of the quark masses in $\overline{MS}$ scheme we use the four-loop approximation and employ the numerical program RunDec described in [39]. The reference value for the quark-gluon coupling is taken as $\alpha_s(m_Z) = 0.1187$ [35]. The alternative choice $\alpha_s(m_\tau) = 0.334$ [35] produces a small difference which we include into the overall counting of uncertainties. We do not attempt to fit the $u$- and $d$-quark masses from the analogous sum rules, and simply take the current (nonlattice) intervals from [35]: $\overline{m}_u(2\,\text{GeV}) = (1.5 - 5.0)$ MeV, $\overline{m}_d(2\,\text{GeV}) = (5.0 - 9.0)$ MeV.

The renormalization scale in our numerical calculation is taken as $\mu = M$, reflecting the average virtuality of perturbative quarks and gluons in the correlator. In order to study the scale dependence we also vary the scale within $M^2/2 < \mu^2 < 2M^2$. The window of Borel parameter is taken as in [18], $2 < M^2 < 3\,\text{GeV}^2$. This choice allows to avoid large nonperturbative effects, simultaneously keeping the excited state contributions reasonably small.

The remaining input parameters used for the OPE of the correlation function are: the quark condensate densities taken from GMOR relation $\langle\bar{u}u\rangle = -f_\pi^2 m_\pi^2/(2(m_u + m_d))$ where $f_\pi = 130.7$ MeV [35]; the ratio of strange and nonstrange condensates $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle = 0.8 \pm 0.3$; the gluon condensate density $\langle\alpha_s/\pi GG\rangle = (0.012^{+0.006}_{-0.012})$ GeV$^4$. Finally, the dimensionful parameter for the quark-gluon condensate density is taken as $m_0^2 = 0.8 \pm 0.2\,\text{GeV}^2$ (for a recent comprehensive review of condensates see [40]).

First of all, we address the main question which interests us here, namely, how good is the convergence of OPE for the correlation function in $N^4LO$, and how large is the numerical impact of the new $O(\alpha_s^4)$ correction. For that we define the ratios:

$$r_n^{(d=0,2)}(M^2) = \frac{\{[\Pi^{(5)''}(M^2)]_{\text{OPE}}^{(d=0,2)}\}^{O(\alpha_s^n)}}{[\Pi^{(5)''}(M^2)]_{\text{OPE}}^{(d=0,2)} + [\Pi^{(5)''}(M^2)]_{\text{OPE}}^{(d=4,6)}} \tag{34}$$

for $n = 0, 1, 2, 3, 4$, where the numerator contains the contribution of $O(\alpha_s^n)$ to the Borel transformed correlation function. The analogous ratio for the nonperturbative contributions is $r^{(d=4,6)}(M^2)$, where the numerator contains only the power suppressed term $[\Pi^{(5)''}(M^2)]_{\text{OPE}}^{(d=4,6)}$. Altogether,

$$\sum_{n=0,1,2,3,4} r_n^{(d=0,2)} + r^{(d=4,6)} = 1.$$

Note that the dominant $m_s$-dependence in $\Pi^{(5)''}(M^2)$ is due to the overall factor $(m_s + m_u)^2$ and largely cancels in $r_n^{(d=0,2)}$ and $r^{(d=4,6)}$. (We use $\overline{m}_s(2\,\text{GeV}) = 105$ MeV in the suppressed terms for this numerical illustration). In Fig. 1 the ratios $r_n^{(d=0,2)}$ and $r^{(d=4,6)}$ are plotted as a function of Borel parameter squared. The convergence is excellent, even beyond the region of the Borel parameter chosen for the sum rule analysis. In the central point $M^2 = 2.5\,\text{GeV}^2$ we obtain $r_n^{(d=0,2)}(2.5\,\text{GeV}^2) =$ 52.4%, 28.3%, 14.4%, 4.0%, −0.3% for $n = 0, 1, 2, 3, 4$, respectively and $r^{(d=4,6)}(2.5\,\text{GeV}^2) = 1.2\%$. We conclude that the currently achieved accuracy of the correlation function at large $M^2$ is quite sufficient for the applications, such as the quark mass determination.

We then turn to the sum rule (6). The input parameters for the kaon contribution to the hadronic part are: $f_K = 159.8$ MeV, $m_K = 493.7$ MeV [35]. The masses and total widths of $K_1$ and $K_2$ resonances [35] were already quoted in the previous section. Their decay constants are taken from [18]: $f_{K_1}/\sqrt{2} = (22.9 \pm 2.4)$ MeV, $f_{K_2}/\sqrt{2} = (14.5 \pm 1.5)$ MeV, where the factor $1/\sqrt{2}$ accounts for the difference between the normalizations. Note that for consistency, we take the version of [18] obtained without ILM correction; furthermore, we added the uncertainties of $f_{K_1}, f_{K_2}$ given in [18] in quadrature. The duality

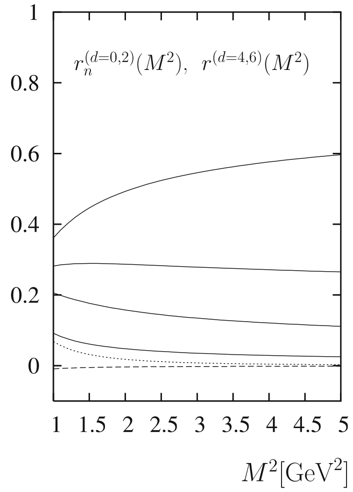


**Fig. 1.** Relative contributions to OPE of the correlation function $\Pi^{(5)''}(M^2)$ defined in (34), plotted as functions of the Borel parameter squared. The *solid lines* from up to down correspond to $r_n^{(d=0,2)}$ with $n = 0, 1, 2, 3$, respectively, the *dashed line* to $r_4^{(d=0,2)}$ and the *dotted line* to $r^{(d=4,6)}$

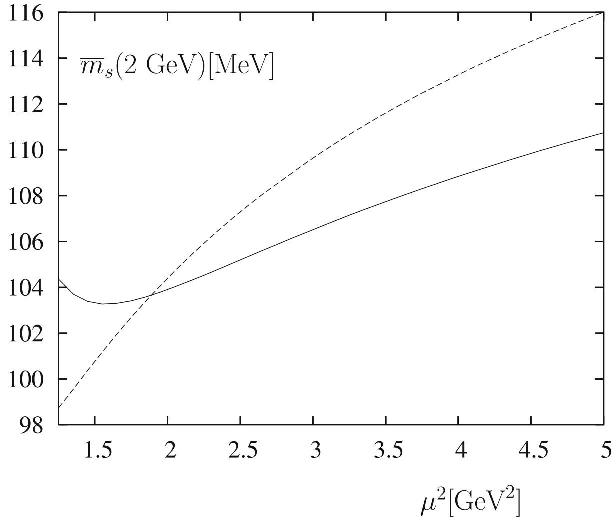


**Fig. 2.** Strange quark mass at the scale 2 GeV, calculated from the sum rule (6) as a function of the renormalization scale $\mu^2$ in the correlation function, varying $\mu^2/M^2$ from 0.5 to 2.0 at $M^2 = 2.5\,\text{GeV}^2$. The *solid* (*dashed*) *line* represents the result obtained with $O(\alpha_s^4)$ ($O(\alpha_s^3)$) accuracy.

threshold adopted in our calculation is $s_0 = 4.5 \pm 0.5\,\text{GeV}^2$. The central value provides the best stability of the sum rule in the Borel parameter interval 2–3 GeV$^2$, whereas the spread of $\pm 0.5\,\text{GeV}^2$ is added to allow for some additional variation of the hadronic input. For the middle values of all parameters specified above, we calculate $\overline{m}_s$ from (6) and obtain the central value presented in (1).

The influence of the new $O(\alpha_s^4)$ correction on the sum rule is estimated by repeating the calculation with the same input, but with the perturbative corrections up to $O(\alpha_s^3)$. The result for the central value turns out to be $\overline{m}_s(2\,\text{GeV}) = 107\,\text{MeV}$, only 2 MeV larger than in (1). We also checked the quality of the OPE in the sum rule. Isolating the OPE part in (6), that is, subtracting from l.h.s. of (6) the integral over $\rho^{(5)}_{\text{OPE}}(s)$ on r.h.s., we calculated the ratios $\tilde{r}_n^{(d=0,2)}(M^2, s_0)$ and $\tilde{r}^{(d=4,6)}(M^2, s_0)$ defined analogous to (34), where instead of $[\Pi^{(5)''}(M^2)]_{\text{OPE}}$ the contributions to the subtracted correlation function

$$M^4[\Pi^{(5)''}(M^2)]_{\text{OPE}} - \int_{s_0}^{\infty} \mathrm{d}s\, e^{-s/M^2} \rho^{(5)}_{\text{OPE}}(s)$$

are substituted. We obtain $\tilde{r}_n^{(d=0,2)}(2.5\,\text{GeV}^2, 4.5\,\text{GeV}^2) = 39.2\%, 26.1\%, 18.8\%, 10.6\%, 3.7\%$ for $n = 0, 1, 2, 3, 4$, respectively, and $\tilde{r}^{(d=4,6)}(2.5\,\text{GeV}^2, 4.5\,\text{GeV}^2) = 1.6\%$ revealing again a good convergence.

Being numerically small, the $O(\alpha_s^4)$ correction is nevertheless important for achieving a better stability with respect to the variation of the renormalization scale $\mu$ entering the sum rule. To demonstrate that, we have calculated $\overline{m}_s(2\,\text{GeV})$ from the sum rules with $O(\alpha_s^3)$ and $O(\alpha_s^4)$ accuracy, varying $\mu^2/M^2$ from 0.5 to 2.0.

The results plotted in Fig. 2 clearly demonstrate the role of the new $O(\alpha_s^4)$ correction in stabilizing the scale-dependence.

To investigate separate theoretical uncertainties of the sum rule (6) in more detail, we group them into two categories:

a) uncertainties related to the input parameters in the correlation function and in the sum rule: renormalization scale, difference between using $\alpha_s(m_z)$ and $\alpha_s(m_\tau)$, Borel parameter, $u$- and $d$-quark masses, condensate densities;

b) uncertainties caused by the hadronic input: the decay constants $f_{K_1}$ and $f_{K_2}$, the effective threshold $s_0$ and the effect of switching off the total widths of $K_{1,2}$.

Varying the input parameters in the QCD part of the sum rule within the limits specified above, we find that the the largest uncertainty in the category (a) is caused by the scale variation (see Fig. 2), whereas the sensitivity to the Borel mass variation is less than $\pm 0.1\,\text{MeV}$, and the dependence on the values of condensate densities is negligible. Adding separate uncertainties grouped in this category in quadrature, we obtain the interval $(\pm 6\,\text{MeV})|_{\text{param}}$ included in (1).

To investigate the hadronic uncertainties grouped above in the category (b), the decay constants $f_{K_1}$, $f_{K_2}$ and the threshold $s_0$ are varied one by one yielding $\pm 5\,\text{MeV}$, $\pm 3\,\text{MeV}$ and $\pm 3\,\text{MeV}$, respectively. To estimate the effect of multiparticle states in the sum rule the $m_s$ calculation is repeated with the total widths of $K_{1,2}$ in (33) set to zero. The result for $\overline{m}_s$ increases by approximately 3 MeV, which we conservatively interprete as an additional uncertainty $\pm 3\,\text{MeV}$. All these individual uncertainties are again added in quadrature to produce $(\pm 7\,\text{MeV})|_{\text{hadr}}$ in (1).

The nonperturbative effects beyond OPE cannot be estimated without the knowledge of the instanton effects, which are beyond our scope and are therefore absent in (1). According to the estimate [18] one has to add $\pm 9\,\text{MeV}$ to the total budget of uncertainties.

**Table 1.** Our estimate of $\overline{m}_s(2\text{GeV})$ compared with some recent determinations obtained with different methods. The error/uncertainty identification in the results taken from the literature can be found in the corresponding papers

| Method | $\overline{m}_s(2\,\text{GeV})$ [MeV] | Ref. |
|---|---|---|
| Pseudoscalar Borel sum | $105 \pm 6 \pm 7$ | This work |
| rule | $100 \pm 6$ | [18](no ILM) |
| Pseudoscalar FESR | $100 \pm 12$ | [18] |
| Scalar Borel sum rule | $99 \pm 16$ | [17] |
| Vector FESR | $139 \pm 31$ | [19] |
| | $81 \pm 22$ | [10] |
| Hadronic $\tau$ decays | $96^{+5+16}_{-3-18}$ | [11] |
| | $104 \pm 28$ | [22] |
| $\tau$ decays $\oplus$ sum rules | $99 \pm 28$ | [22] |
| | $97 \pm 22$ | [2] |
| Lattice QCD ($n_f = 2$) | 100–130 | [4] |
| | $101 \pm 8^{+25}_{-0}$ | [5] |
| | $76 \pm 3 \pm 7$ | [1] |
| Lattice QCD ($n_f = 3$) | $86.7 \pm 5.9$ | [3] |
| | $87 \pm 4 \pm 4$ | [6] |
| PDG04 average | 80–130 | [35] |

In Table 1 we compare our prediction with other determinations of $m_s$. The values of the $s$ quark mass obtained from the correlation functions ($\tau$ decays, Borel sum rules and FESR ) are consistent with each other and with the lattice QCD results within still large uncertainties, the lattice results with $n_f = 3$ being systematically lower. Our estimate (1) is also consistent with the $s$-quark mass bound in $O(\alpha_s^4)$ obtained in [23].

## 6 Conclusion

We have included the new $O(\alpha_s^4)$ correction in the correlation function of the pseudoscalar strangeness-changing quark currents and calculated the $s$-quark mass from the resulting Borel sum rule. In future the same analysis should be repeated for the scalar Borel sum rule, for both pseudoscalar and scalar FESR and for the sum rules with non-strange light-quark currents, yielding the $u$ and $d$ quark masses.

Our main intention here was to investigate the role of the $O(\alpha_s^4)$ terms in the OPE and in the sum rule. We have found that the new correction is comfortably small, making OPE in this channel very reliable. Simultaneously, the addition of the $O(\alpha_s^4)$ contributions noticeably decreases the renormalization scale-dependence of the resulting sum rule.

The QCD sum rules obtained on the basis of OPE still have a considerable room of improvement. While the nonperturbative effects beyond OPE can be kept under control by choosing the virtuality (Borel parameter) scale large enough, and using ILM-type estimates, there is still a lack of experimental information concerning the masses, total and partial widths of the excited kaon resonances. The resonance $K_1$ can be observed in $\tau \to K\pi\pi\nu_\tau$ decays, and both $K_1$ and $K_2$ probably also in hadronic $B$ decays where the currently available statistics allows to isolate many light-quark resonances in the final states. With this information one would be able to built a hadronic spectral function in the pseudoscalar channel which is less dependent on duality ansatz, so that the accuracy of the hadronic part of the sum rule eventually becomes closer to the high precision achieved in the QCD part.

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